Zoli's arborescence packing theorems via matroid intersection

Csaba Király[1] Zoltán Szigeti[2] Shin-ichi Tanigawa[3]

[1]MTA-ELTE Egerváry Research Group on Combinatorial Optimization, and Dept. of Operations Research, ELTE Eötvös Loránd University, Budapest, Hungary

[2]Univ. Grenoble Alpes, G-SCOP, Grenoble, France

[3]Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo Tokyo, Japan

Combinatorial Optimization Day: Orientations, Matchings and Rigidity
2018. 03. 20.

packing = edge-disjoint subgraphs

Theorem (Tutte, Nash-Williams)

In a graph $D = (V, E)$, there exists a packing of k spanning trees iff

$$e_G(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$$

holds for every partition $\mathcal{P}$ of V, where $e_G(\mathcal{P})$ denotes the number of edges that are not induced by any set of the partition.

Matroid structure behind

graphic matroid = independent sets are edge sets of forests in a graph
k-sum of $\mathcal{M}$ = independent sets can be partitioned into k independent sets of $\mathcal{M}$.
Union of k spanning trees = bases of the k-sum of the graphic matroid.

packing = edge-disjoint subgraphs

Theorem (Tutte, Nash-Williams)

In a graph $D = (V, E)$, there exists a packing of k spanning trees iff

$$e_G(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$$

holds for every partition $\mathcal{P}$ of V, where $e_G(\mathcal{P})$ denotes the number of edges that are not induced by any set of the partition.

Matroid structure behind

graphic matroid = independent sets are edge sets of forests in a graph
k-sum of $\mathcal{M}$ = independent sets can be partitioned into k independent sets of $\mathcal{M}$.
Union of k spanning trees = bases of the k-sum of the graphic matroid.

packing = edge-disjoint subgraphs

Theorem (Tutte, Nash-Williams)

In a graph $D = (V, E)$, there exists a packing of k spanning trees iff

$$e_G(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$$

holds for every partition $\mathcal{P}$ of V, where $e_G(\mathcal{P})$ denotes the number of edges that are not induced by any set of the partition.

Matroid structure behind

graphic matroid = independent sets are edge sets of forests in a graph
k-sum of $\mathcal{M}$ = independent sets can be partitioned into k independent sets of $\mathcal{M}$.
Union of k spanning trees = bases of the k-sum of the graphic matroid.

s-arborescence = directed tree s.t. each node is reachable from its root on a one-way path

ϱ = the in-degree

Theorem (Edmonds)

In a rooted digraph $D = (V + s, A)$, there exists a packing of k spanning s-arborescences iff

$$\varrho(X) \geq k$$

holds for every $\emptyset \neq X \subseteq V$.

s-arborescence = directed tree s.t. each node is reachable from its root on a one-way path

ϱ = the in-degree

Theorem (Edmonds)

In a rooted digraph $D = (V + s, A)$, there exists a packing of k spanning s-arborescences iff

$$\varrho(X) \geq k$$

holds for every $\emptyset \neq X \subseteq V$.

s-arborescence = directed tree s.t. each node is reachable from its root on a one-way path

ϱ = the in-degree

Theorem (Edmonds)

In a rooted digraph $D = (V + s, A)$, there exists a packing of k spanning s-arborescences iff

$$\varrho(X) \geq k$$

holds for every $\emptyset \neq X \subseteq V$.

Remark

The Tutte–Nash-Williams-theorem follows from Edmonds' theorem by using Frank's orientation theorems.

s-arborescence = directed tree s.t. each node is reachable from its root on a one-way path

ϱ = the in-degree

Theorem (Edmonds)

In a rooted digraph $D = (V + s, A)$, there exists a packing of k spanning s-arborescences iff

$$\varrho(X) \geq k$$

holds for every $\emptyset \neq X \subseteq V$.

Matroid structure behind

Let the independent sets of $\mathcal{M}_0$ be the arc sets of D with maximum in-degree k on V = direct sum of the uniform matroids of rank k on the incoming arc sets of all vertices in V.
Union of k spanning arborescences = common bases of the k-sum of the graphic matroid and $\mathcal{M}_0$.

s-arborescence = directed tree s.t. each node is reachable from its root on a one-way path

ϱ = the in-degree

Theorem (Edmonds)

In a rooted digraph $D = (V + s, A)$, there exists a packing of k spanning s-arborescences iff

$$\varrho(X) \geq k$$

holds for every $\emptyset \neq X \subseteq V$.

Matroid structure behind

Let the independent sets of $\mathcal{M}_0$ be the arc sets of D with maximum in-degree k on V = direct sum of the uniform matroids of rank k on the incoming arc sets of all vertices in V.
Union of k spanning arborescences = common bases of the k-sum of the graphic matroid and $\mathcal{M}_0$. $\Rightarrow$ Efficient algorithm for the weighted

Let $\mathcal{M}_2$ be the direct sum of matroids $\mathcal{M}_v$ on the incoming arc sets of all $v \in V$, we get the following.
$\mathcal{M}_2$-restricted packing = the arc set is independent in $\mathcal{M}_2$.
$\partial_Z(X)$ = the arc set that enters X from $Z - X$.

Theorem (Frank)

In a rooted digraph $D = (V + s, A)$ with a matroid $\mathcal{M}_2 = \oplus_{v \in V} \mathcal{M}_v$ on A as above, there exists an $\mathcal{M}_2$-restricted packing of k spanning s-arborescences iff

$$r_2(\partial_{V+s}(X)) \geq k$$

holds for every $\emptyset \neq X \subseteq V$.

Proof.

Change $\mathcal{M}_0$ to $\mathcal{M}_2$ in the previous formulation and use Edmonds' theorem. □

Let $\mathcal{M}_2$ be the direct sum of matroids $\mathcal{M}_v$ on the incoming arc sets of all $v \in V$, we get the following.
$\mathcal{M}_2$-restricted packing = the arc set is independent in $\mathcal{M}_2$.
$\partial_Z(X)$ = the arc set that enters X from $Z - X$.

Theorem (Frank)

In a rooted digraph $D = (V + s, A)$ with a matroid $\mathcal{M}_2 = \oplus_{v \in V} \mathcal{M}_v$ on A as above, there exists an $\mathcal{M}_2$-restricted packing of k spanning s-arborescences iff

$$r_2(\partial_{V+s}(X)) \geq k$$

holds for every $\emptyset \neq X \subseteq V$.

Proof.

Change $\mathcal{M}_0$ to $\mathcal{M}_2$ in the previous formulation and use Edmonds' theorem. □

Kamiyama, Katoh, Takizawa: If there are no spanning arborescences... When is it possible to pack edge-disjoint “maximal” arborescences?

Reachability s-arborescence in D: an s-arborescence that spans each vertex which is reachable from s on a one-way path of D.

Theorem (Kamiyama, Katoh and Takizawa)

In a digraph $D = (V, A)$, let $R := \{s_1, ..., s_k\}$ be a multiset of vertices in V. There exists a packing of reachability s_i-arborescences in D $(i = 1, \dots, k)$ iff

$$\varrho(X) \geq p'_R(X)$$

holds for every $X \subseteq V$ where $p'_R(X)$ denotes the number of s_i's for which X is reachable from s_i and $s_i \notin X$.

Kamiyama, Katoh, Takizawa: If there are no spanning arborescences... When is it possible to pack edge-disjoint "maximal" arborescences?

Reachability s-arborescence in D: an s-arborescence that spans each vertex which is reachable from s on a one-way path of D.

Theorem (Kamiyama, Katoh and Takizawa)

In a digraph $D = (V, A)$, let $R := \{s_1, ..., s_k\}$ be a multiset of vertices in V. There exists a packing of reachability s_i-arborescences in D $(i = 1, \ldots, k)$ iff

$$\varrho(X) \geq p'_R(X)$$

holds for every $X \subseteq V$ where $p'_R(X)$ denotes the number of s_i's for which X is reachable from s_i and $s_i \notin X$.

Definition

1. matroid-rooted graph/digraph $(G = (V + s, E), \mathcal{M}_1)$/ $(D = (V + s, A), \mathcal{M}_1)$: a matroid $\mathcal{M}_1$ is given on the set of root edges/arcs (leaving s).

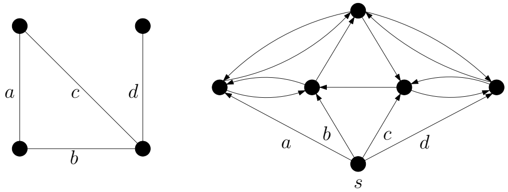


Figure: a matroid-rooted digraph

Definition

1. matroid-rooted graph/digraph $(G = (V + s, E), \mathcal{M}_1)$/ $(D = (V + s, A), \mathcal{M}_1)$: a matroid $\mathcal{M}_1$ is given on the set of root edges/arcs (leaving s).
2. $\mathcal{M}_1$-based packing of (s, t)-paths: if the root edges/arcs form a base of $\mathcal{M}_1$.

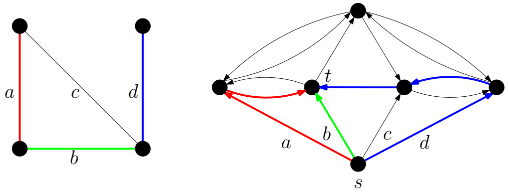


Figure: an $\mathcal{M}_1$-based packing of (s, t)-paths

Definition

1. matroid-rooted graph/digraph $(G = (V + s, E), \mathcal{M}_1)$/ $(D = (V + s, A), \mathcal{M}_1)$: a matroid $\mathcal{M}_1$ is given on the set of root edges/arcs (leaving s).
2. $\mathcal{M}_1$-based packing of (s, t)-paths: if the root edges/arcs form a base of $\mathcal{M}_1$.

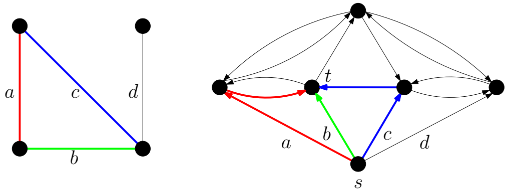


Figure: Not an $\mathcal{M}_1$-based packing of (s, t)-paths

Definition

1. matroid-rooted graph/digraph $(G = (V+s, E), \mathcal{M}_1)$/ $(D = (V+s, A), \mathcal{M}_1)$: a matroid $\mathcal{M}_1$ is given on the set of root edges/arcs (leaving s).
2. $\mathcal{M}_1$-based packing of (s,t)-paths: if the root edges/arcs form a base of $\mathcal{M}_1$.
3. $\mathcal{M}_1$-based packing of s-trees/arborescences: if the packing of (s,t)-paths provided by the trees/arborescences is $\mathcal{M}_1$-based $\forall t \in V$.

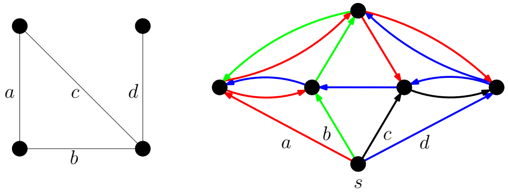


Figure: an $\mathcal{M}_1$-based packing of s-arborescences

Definition

1. matroid-rooted graph/digraph $(G=(V+s,E),\mathcal{M}_1)$/ $(D=(V+s,A),\mathcal{M}_1)$: a matroid $\mathcal{M}_1$ is given on the set of root edges/arcs (leaving s).
2. $\mathcal{M}_1$-based packing of (s,t)-paths: if the root edges/arcs form a base of $\mathcal{M}_1$.
3. $\mathcal{M}_1$-based packing of s-trees/arborescences: if the packing of (s,t)-paths provided by the trees/arborescences is $\mathcal{M}_1$-based $\forall t \in V$.

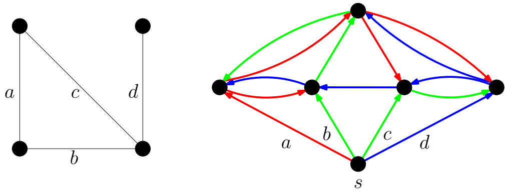


Figure: an $\mathcal{M}_1$-based packing of spanning s-arborescences

Definition

1. matroid-rooted graph/digraph $(G = (V + s, E), \mathcal{M}_1)$/ $(D = (V + s, A), \mathcal{M}_1)$: a matroid $\mathcal{M}_1$ is given on the set of root edges/arcs (leaving s).
2. $\mathcal{M}_1$-based packing of (s, t)-paths: if the root edges/arcs form a base of $\mathcal{M}_1$.
3. $\mathcal{M}_1$-based packing of s-trees/arborescences: if the packing of (s, t)-paths provided by the trees/arborescences is $\mathcal{M}_1$-based $\forall t \in V$.

Remark

Menger type characterization: $\exists$ an $\mathcal{M}_1$-based packing of (s, t)-paths iff $\varrho(X) \geq r_1(\partial_s(V)) - r_1(\partial_s(X))$ $(\forall X \subseteq V)$.

Definition

1. matroid-rooted graph/digraph $(G = (V + s, E), \mathcal{M}_1)$/ $(D = (V + s, A), \mathcal{M}_1)$: a matroid $\mathcal{M}_1$ is given on the set of root edges/arcs (leaving s).
2. $\mathcal{M}_1$-based packing of (s, t)-paths: if the root edges/arcs form a base of $\mathcal{M}_1$.
3. $\mathcal{M}_1$-based packing of s-trees/arborescences: if the packing of (s, t)-paths provided by the trees/arborescences is $\mathcal{M}_1$-based $\forall t \in V$.

Remark

Menger type characterization: $\exists$ an $\mathcal{M}_1$-based packing of (s, t)-paths iff $\varrho(X) \geq r_1(\partial_s(V)) - r_1(\partial_s(X))$ $(\forall X \subseteq V)$.

Question

Can the above theorems be extended for $\mathcal{M}_1$-based packings?

Theorem (Katoh, Tanigawa)

In a matroid-rooted graph $(G = (V + s, E), \mathcal{M})$ there exists an $\mathcal{M}_1$-based packing of ~~spanning~~ s-trees iff

$$e_G(\mathcal{P}) \geq r_1(\partial_s(V)) - \sum_{X \in \mathcal{P}} r_1(\partial_s(X)) \text{ for every partition } \mathcal{P} \text{ of } V.$$

Matroid structure behind

Katoh and Tanigawa also proved that the $\mathcal{M}_1$-based packings of s-trees form the bases of the matroid induced by the following non-negative integer valued, monotone and intersecting submodular function:

$$b'(H) := k|V(H) - s| - k + r_1(H \cap \partial_s(V)) \ \ \forall \emptyset \neq H \subseteq A,$$

i.e. the matroid $\mathcal{M}_{b'}$ with independent sets

$$\mathcal{I}_{b'} := \{B \subseteq A : |H| \leq b'(H) \ \forall \emptyset \neq H \subseteq B\}.$$

Theorem (Katoh, Tanigawa)

In a matroid-rooted graph $(G = (V + s, E), \mathcal{M})$ there exists an $\mathcal{M}_1$-based packing of ~~spanning~~ s-trees iff

$$e_G(\mathcal{P}) \geq r_1(\partial_s(V)) - \sum_{X \in \mathcal{P}} r_1(\partial_s(X)) \text{ for every partition } \mathcal{P} \text{ of } V.$$

Remark

Motivation from rigidity theory! (Title of this workshop: Combinatorial Optimization Day: Orientations, Matchings and Rigidity.)

Matroid structure behind

Katoh and Tanigawa also proved that the $\mathcal{M}_1$-based packings of s-trees form the bases of the matroid induced by the following non-negative integer valued, monotone and intersecting submodular function:

$$b'(H) := k|V(H) - s| - k + r_1(H \cap \partial_s(V)) \quad \forall \emptyset \neq H \subseteq A,$$

Theorem (Katoh, Tanigawa)

In a matroid-rooted graph $(G = (V + s, E), \mathcal{M})$ *there exists an* $\mathcal{M}_1$*-based packing of ~~spanning~~ s-trees iff*

$$e_G(\mathcal{P}) \geq r_1(\partial_s(V)) - \sum_{X \in \mathcal{P}} r_1(\partial_s(X)) \text{ for every partition } \mathcal{P} \text{ of } V.$$

Matroid structure behind

Katoh and Tanigawa also proved that the $\mathcal{M}_1$-based packings of s-trees form the bases of the matroid induced by the following non-negative integer valued, monotone and intersecting submodular function:

$$b'(H) := k|V(H) - s| - k + r_1(H \cap \partial_s(V)) \quad \forall \emptyset \neq H \subseteq A,$$

i.e. the matroid $\mathcal{M}_{b'}$ with independent sets

$$\mathcal{I}_{b'} := \{B \subseteq A : |H| \leq b'(H) \; \forall \emptyset \neq H \subseteq B\}.$$

Theorem (Durand de Gevigney, Nguyen and Szigeti)

In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M}_1)$ there exists an $\mathcal{M}_1$-based packing of ~~spanning~~ s-arborescences iff there exists an $\mathcal{M}_1$-based packing of (s, v)-paths for every $v \in V$, i.e.

$$\varrho(X) \geq r_1(\mathcal{M}) - r_1(\partial_s(X)) \ (\forall \emptyset \neq X \subseteq V).$$

Remark

The Katoh–Tanigawa-theorem follows from this theorem by using Frank's orientation theorems.

Matroid structure behind

Edge sets of $\mathcal{M}_1$-based packing of s-arborescences = common bases of $\mathcal{M}_{b'}$ and $\mathcal{M}_0$.

Theorem (Durand de Gevigney, Nguyen and Szigeti)

In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M}_1)$ there exists an $\mathcal{M}_1$-based packing of ~~spanning~~ s-arborescences iff there exists an $\mathcal{M}_1$-based packing of (s, v)-paths for every $v \in V$, i.e.

$$\varrho(X) \geq r_1(\mathcal{M}) - r_1(\partial_s(X)) \ (\forall \emptyset \neq X \subseteq V).$$

Remark

The Katoh–Tanigawa-theorem follows from this theorem by using Frank's orientation theorems.

Matroid structure behind

Edge sets of $\mathcal{M}_1$-based packing of s-arborescences = common bases of $\mathcal{M}_{b'}$ and $\mathcal{M}_0$.

Theorem (Durand de Gevigney, Nguyen and Szigeti)

In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M}_1)$ *there exists an* $\mathcal{M}_1$*-based packing of* ~~*spanning*~~ *s-arborescences iff there exists an* $\mathcal{M}_1$*-based packing of* (s, v)*-paths for every* $v \in V$*, i.e.*

$$\varrho(X) \geq r_1(\mathcal{M}) - r_1(\partial_s(X)) \ (\forall \emptyset \neq X \subseteq V).$$

Remark

The Katoh–Tanigawa-theorem follows from this theorem by using Frank's orientation theorems.

Matroid structure behind

Edge sets of $\mathcal{M}_1$-based packing of s-arborescences = common bases of $\mathcal{M}_{b'}$ and $\mathcal{M}_0$.

Theorem

In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M}_1)$ with another matroid $\mathcal{M}_2 = \oplus_{v \in V} \mathcal{M}_v$ on A, there exists an $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of ~~spanning~~ s-arborescences iff

$$r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial_s(V)) \text{ for all } \emptyset \neq X \subseteq V \text{ and } F \subseteq \partial_s(X).$$

Proof.

Change $\mathcal{M}_0$ to $\mathcal{M}_2$ in the previous formulation and use the Durand de Gevigney–Nguyen–Szigeti-theorem. □

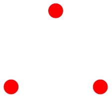
Conjecture (Bérczi, Frank, T. Király, Kobayashi)

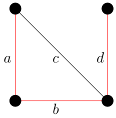
In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M}_1)$ there exists an $\mathcal{M}_1$-based packing of spanning s-arborescences iff

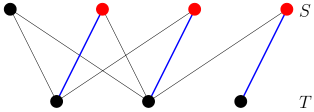
$$\varrho(X) \geq r_1(\mathcal{M}) - r_1(\partial_s(X)) \ (\forall \emptyset \neq X \subseteq V).$$

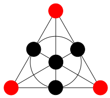
Examples

1. Free : all subsets of a set,
2. Graphic : edge-sets of forests of a graph,
3. Transversal: end-nodes in S of matchings of bipartite graph $(S, T; E)$
4. Fano: subsets of sets of size 3 not being a line in the Fano plane.









Conjecture (Bérczi, Frank, T. Király, Kobayashi)

In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M}_1)$ there exists an $\mathcal{M}_1$-based packing of spanning s-arborescences iff

$$\varrho(X) \geq r_1(\mathcal{M}) - r_1(\partial_s(X)) \ (\forall \emptyset \neq X \subseteq V).$$

Theorem (Fortier, K., Szigeti, Tanigawa)

The conjecture is true when the matroid $\mathcal{M}_1$

- *has rank at most 2 or*
- *is graphic or*
- *is transversal.*

The conjecture is false!
The corresponding decision problem is NP-hard.

Conjecture (Bérczi, Frank, T. Király, Kobayashi)

In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M}_1)$ there exists an $\mathcal{M}_1$-based packing of spanning s-arborescences iff

$$\varrho(X) \geq r_1(\mathcal{M}) - r_1(\partial_s(X)) \; (\forall \emptyset \neq X \subseteq V).$$

Theorem (Fortier, K., Szigeti, Tanigawa)

The conjecture is true when the matroid $\mathcal{M}_1$

- *has rank at most 2 or*
- *is graphic or*
- *is transversal.*

The conjecture is false!
The corresponding decision problem is NP-hard.

Conjecture (Bérczi, Frank, T. Király, Kobayashi)

In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M}_1)$ there exists an $\mathcal{M}_1$-based packing of spanning s-arborescences iff

$$\varrho(X) \geq r_1(\mathcal{M}) - r_1(\partial_s(X)) \ (\forall \emptyset \neq X \subseteq V).$$

Theorem (Fortier, K., Szigeti, Tanigawa)

The conjecture is true when the matroid $\mathcal{M}_1$

- *has rank at most 2 or*
- *is graphic or*
- *is transversal.*

The conjecture is false!

The corresponding decision problem is NP-hard.

Conjecture (Bérczi, Frank, T. Király, Kobayashi)

In a matroid-rooted digraph $(D=(V+s,A),\mathcal{M}_1)$ there exists an $\mathcal{M}_1$-based packing of spanning s-arborescences iff

$$\varrho(X)\geq r_1(\mathcal{M})-r_1(\partial_s(X))\ (\forall\emptyset\neq X\subseteq V).$$

Theorem (Fortier, K., Szigeti, Tanigawa)

The conjecture is true when the matroid $\mathcal{M}_1$

- *has rank at most 2 or*
- *is graphic or*
- *is transversal.*

The conjecture is false!
The corresponding decision problem is NP-hard.

Conjecture (Bérczi, Frank, T. Király, Kobayashi)

In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M}_1)$ there exists an $\mathcal{M}_1$-based packing of spanning s-arborescences iff

$$\varrho(X) \geq r_1(\mathcal{M}) - r_1(\partial_s(X)) \ (\forall \emptyset \neq X \subseteq V).$$

Theorem (Fortier, K., Szigeti, Tanigawa)

The conjecture is true when the matroid $\mathcal{M}_1$

- *has rank at most 2 or*
- *is graphic or*
- *is transversal.*

The conjecture is false!

The corresponding decision problem is NP-hard.

Conjecture (Bérczi, Frank, T. Király, Kobayashi)

In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M}_1)$ there exists an $\mathcal{M}_1$-based packing of spanning s-arborescences iff

$$\varrho(X) \geq r_1(\mathcal{M}) - r_1(\partial_s(X)) \ (\forall \emptyset \neq X \subseteq V).$$

Theorem (Fortier, K., Szigeti, Tanigawa)

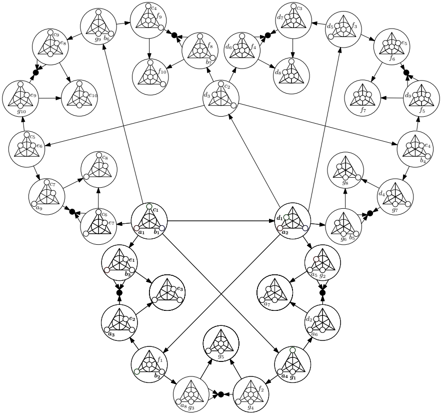
The conjecture is true when the matroid $\mathcal{M}_1$

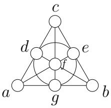
- *has rank at most 2 or*
- *is graphic or*
- *is transversal.*

The conjecture is false!
The corresponding decision problem is NP-hard.

Counterexample

Digraph : acyclic, in-degree 3 for all $v \in V$, 46 nodes and 135 arcs,
Matroid : parallel extension of Fano with 64 elements,
Remark : matroid-based packing of (s, t)-paths exists for all t.





Theorem (Fortier, K., Szigeti, Tanigawa)

The conjecture is true when the matroid $\mathcal{M}_1$

- *has rank at most 2 or*
- *is graphic or*
- *is transversal.*

The conjecture is false!
The corresponding decision problem is NP-hard.

Remarks

- $\mathcal{M}_1$-based packing of spanning s-trees: same results by orientation and the structure of the counterexample
- $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of spanning s-arborescences: same positive results

Theorem (Fortier, K., Szigeti, Tanigawa)

The conjecture is true when the matroid $\mathcal{M}_1$

- *has rank at most 2 or*
- *is graphic or*
- *is transversal.*

The conjecture is false!
The corresponding decision problem is NP-hard.

Remarks

- $\mathcal{M}_1$-based packing of spanning s-trees: same results by orientation and the structure of the counterexample
- $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of spanning s-arborescences: same positive results

$P(X) = \{v \in V : \exists$ a one-way path from v to $X\}$. ($X \subseteq P(X)$)

$\mathcal{M}_1$-reachability-based packing of (s,t)-paths: if the root arcs form a base of $\mathcal{M}_1|_{\partial_s(P(t))}$.

$\mathcal{M}_1$-reachability-based packing of s-arborescences: if the packing of (s,t)-paths provided by the arborescences is $\mathcal{M}_1$-reachability-based $\forall t \in V$.

Theorem (K.)

In a matroid-rooted digraph $(D = (V+s, A), \mathcal{M}_1)$ there exists an $\mathcal{M}_1$-reachability-based packing of s-arborescences iff

$$\varrho(X) \geq r_1(\partial_s(P(X))) - r_1(\partial_s(X)) \; (\forall X \subseteq V).$$

Definitions

Biset $X = (X_O, X_I)$: $X_I \subseteq X_O \subseteq V$

$\mathcal{P}_2$ = all bisets on V

$X \cap Y = (X_O \cap Y_O, X_I \cap Y_I)$

$X \cup Y = (X_O \cup Y_O, X_I \cup Y_I)$

X and Y are intersecting = $X_I \cap Y_I \neq \emptyset$

$b : \mathcal{P}_2 \to \mathbb{Z}_+ \cup \{\infty\}$ is intersecting submodular =

$b(X) + b(Y) \geq b(X \cup Y) + b(X \cap Y)$ for every intersecting $X, Y \in \mathcal{P}_2$.

$D = (V, A)$, $B \subseteq A$, $X \in \mathcal{P}_2$.

$B(X)$ = arcs in B with tail in X_O and head in X_I.

$i_B(X) = |B(X)|$

Definitions

Biset $X = (X_O, X_I)$: $X_I \subseteq X_O \subseteq V$
$\mathcal{P}_2$ = all bisets on V
$X \cap Y = (X_O \cap Y_O, X_I \cap Y_I)$
$X \cup Y = (X_O \cup Y_O, X_I \cup Y_I)$
X and Y are intersecting = $X_I \cap Y_I \neq \emptyset$
$b : \mathcal{P}_2 \to \mathbb{Z}_+ \cup \{\infty\}$ is intersecting submodular =
$b(X) + b(Y) \geq b(X \cup Y) + b(X \cap Y)$ for every intersecting $X, Y \in \mathcal{P}_2$.
$D = (V, A)$, $B \subseteq A$, $X \in \mathcal{P}_2$.
$B(X)$ = arcs in B with tail in X_O and head in X_I.
$i_B(X) = |B(X)|$

Definitions

Biset $X = (X_O, X_I)$: $X_I \subseteq X_O \subseteq V$
$\mathcal{P}_2$ = all bisets on V
$X \cap Y = (X_O \cap Y_O, X_I \cap Y_I)$
$X \cup Y = (X_O \cup Y_O, X_I \cup Y_I)$
X and Y are intersecting = $X_I \cap Y_I \neq \emptyset$
$b : \mathcal{P}_2 \to \mathbb{Z}_+ \cup \{\infty\}$ is intersecting submodular =
$b(X) + b(Y) \geq b(X \cup Y) + b(X \cap Y)$ for every intersecting $X, Y \in \mathcal{P}_2$.
$D = (V, A)$, $B \subseteq A$, $X \in \mathcal{P}_2$.
$B(X)$ = arcs in B with tail in X_O and head in X_I.
$i_B(X) = |B(X)|$

Definitions

Biset $X = (X_O, X_I)$: $X_I \subseteq X_O \subseteq V$
$\mathcal{P}_2$ = all bisets on V
$X \cap Y = (X_O \cap Y_O, X_I \cap Y_I)$
$X \cup Y = (X_O \cup Y_O, X_I \cup Y_I)$
X and Y are intersecting = $X_I \cap Y_I \neq \emptyset$
$b : \mathcal{P}_2 \to \mathbb{Z}_+ \cup \{\infty\}$ is intersecting submodular =
$b(X) + b(Y) \geq b(X \cup Y) + b(X \cap Y)$ for every intersecting $X, Y \in \mathcal{P}_2$.
$D = (V, A)$, $B \subseteq A$, $X \in \mathcal{P}_2$.
$B(X)$ = arcs in B with tail in X_O and head in X_I.
$i_B(X) = |B(X)|$

Definitions

Biset $X = (X_O, X_I)$: $X_I \subseteq X_O \subseteq V$
$\mathcal{P}_2$ = all bisets on V
$X \cap Y$ = $(X_O \cap Y_O, X_I \cap Y_I)$
$X \cup Y$ = $(X_O \cup Y_O, X_I \cup Y_I)$
X and Y are intersecting = $X_I \cap Y_I \neq \emptyset$
$b : \mathcal{P}_2 \to \mathbb{Z}_+ \cup \{\infty\}$ is intersecting submodular =
$b(X) + b(Y) \geq b(X \cup Y) + b(X \cap Y)$ for every intersecting $X, Y \in \mathcal{P}_2$.
$D = (V, A)$, $B \subseteq A$, $X \in \mathcal{P}_2$.
$B(X)$ = arcs in B with tail in X_O and head in X_I.
$i_B(X)$ = $|B(X)|$

Matroids from submodular bi-set functions

Theorem

Let $D = (V, A)$ *be a digraph and* $b : \mathcal{P}_2 \to \mathbb{Z}_+ \cup \{\infty\}$ *an intersecting submodular bi-set function. Then*

$$\mathcal{I} := \{B \subseteq A : i_B(\mathsf{X}) \leq \mathsf{b}(\mathsf{X})\ \forall \mathsf{X} \in \mathcal{P}_2\}$$

forms the family of independent sets of a matroid $\mathcal{M}_b$ *on* A*.*

Matroid structure behind reachability-based packings

We give the construction directly for $\mathcal{M}_2$-restricted $\mathcal{M}_1$-reachability-based packings.

Assumptions

(A1) $\partial_s(v)$ is independent in $\mathcal{M}_1$ for every $v \in V$;

(A2) each root arc belongs to every base of $\mathcal{M}_2$;

(A3) $r_2(\partial(v)) \leq r_1(\partial_s(P(v)))$ for all $v \in V$.

These can be assumed by simple modifications of the input (adding an extra vertex in the middle of each root arc and truncating $\mathcal{M}_2$.)
When the packing exists, in fact, (A3) turns to

(A3') $r_2(\partial(v)) = r_1(\partial_s(P(v))) =: m(v)$ for all $v \in V$.

Matroid structure behind reachability-based packings

We give the construction directly for $\mathcal{M}_2$-restricted $\mathcal{M}_1$-reachability-based packings.

Assumptions

(A1) $\partial_s(v)$ is independent in $\mathcal{M}_1$ for every $v \in V$;

(A2) each root arc belongs to every base of $\mathcal{M}_2$;

(A3) $r_2(\partial(v)) \leq r_1(\partial_s(P(v)))$ for all $v \in V$.

These can be assumed by simple modifications of the input (adding an extra vertex in the middle of each root arc and truncating $\mathcal{M}_2$.)

When the packing exists, in fact, (A3) turns to

(A3') $r_2(\partial(v)) = r_1(\partial_s(P(v))) =: m(v)$ for all $v \in V$.

Matroid structure behind reachability-based packings

We give the construction directly for $\mathcal{M}_2$-restricted $\mathcal{M}_1$-reachability-based packings.

Assumptions

(A1) $\partial_s(v)$ is independent in $\mathcal{M}_1$ for every $v \in V$;

(A2) each root arc belongs to every base of $\mathcal{M}_2$;

(A3) $r_2(\partial(v)) \leq r_1(\partial_s(P(v)))$ for all $v \in V$.

These can be assumed by simple modifications of the input (adding an extra vertex in the middle of each root arc and truncating $\mathcal{M}_2$.)

When the packing exists, in fact, (A3) turns to

(A3') $r_2(\partial(v)) = r_1(\partial_s(P(v))) =: m(v)$ for all $v \in V$.

Matroid structure behind reachability-based packings

We give the construction directly for $\mathcal{M}_2$-restricted $\mathcal{M}_1$-reachability-based packings.

Assumptions

- (A1) $\partial_s(v)$ is independent in $\mathcal{M}_1$ for every $v \in V$;
- (A2) each root arc belongs to every base of $\mathcal{M}_2$;
- (A3) $r_2(\partial(v)) \leq r_1(\partial_s(P(v)))$ for all $v \in V$.

These can be assumed by simple modifications of the input (adding an extra vertex in the middle of each root arc and truncating $\mathcal{M}_2$.)
When the packing exists, in fact, (A3) turns to

(A3') $r_2(\partial(v)) = r_1(\partial_s(P(v))) =: m(v)$ for all $v \in V$.

Matroid structure behind reachability-based packings

$u \sim v$ = $P(u) = P(v)$

Atoms = equivalence classes of $\sim$

$$
\begin{aligned}
\mathcal{F} &:= \{X \in \mathcal{P}_2 : \exists \text{ atom } A : \emptyset \neq X_I \subseteq A, (X_O \setminus X_I) \cap A = \emptyset\}, \\
I_X &:= \{e_i \in \partial_s^A(V) : X_I \subseteq U_i, e_i \notin \partial_s^A(X_I), (X_O \setminus X_I) \cap U_i = \emptyset\} && (\forall X \in \mathcal{F}), \\
J_X &:= \{e_i \in \partial_s^A(V) : X_I \subseteq U_i\} \setminus I_X && (\forall X \in \mathcal{F}), \\
b(X) &:= \widetilde{m}(X_I) - |\partial_s^A(X_I)| - r_1(I_X \cup J_X) + r_1(J_X) && (\forall X \in \mathcal{F}), \\
&:= +\infty && (\forall X \notin \mathcal{F}).
\end{aligned}
$$

Remark

$$I_X \cup J_X = \partial_s(P(X_I)).$$

Matroid structure behind reachability-based packings

$u \sim v$ = $P(u) = P(v)$

Atoms = equivalence classes of $\sim$

$$\mathcal{F} := \{X \in \mathcal{P}_2 : \exists \text{ atom } A : \emptyset \neq X_I \subseteq A, (X_O \setminus X_I) \cap A = \emptyset\},$$

$$I_X := \{e_i \in \partial_s^A(V) : X_I \subseteq U_i, e_i \notin \partial_s^A(X_I), (X_O \setminus X_I) \cap U_i = \emptyset\} \quad (\forall X \in \mathcal{F}),$$

$$J_X := \{e_i \in \partial_s^A(V) : X_I \subseteq U_i\} \setminus I_X \quad (\forall X \in \mathcal{F}),$$

$$b(X) := \widetilde{m}(X_I) - |\partial_s^A(X_I)| - r_1(I_X \cup J_X) + r_1(J_X) \quad (\forall X \in \mathcal{F}),$$

$$:= +\infty \quad (\forall X \notin \mathcal{F}).$$

Remark

$$I_X \cup J_X = \partial_s(P(X_I)).$$

Matroid structure behind reachability-based packings

$u{\sim}v = P(u) = P(v)$
Atoms = equivalence classes of $\sim$

$$\begin{aligned}
\mathcal{F} &:= \{\mathrm{X} \in \mathcal{P}_2 : \exists \text{ atom } \mathrm{A} : \emptyset \neq \mathrm{X_I} \subseteq \mathrm{A}, (\mathrm{X_O} \setminus \mathrm{X_I}) \cap \mathrm{A} = \emptyset\}, \\
I_\mathrm{X} &:= \{e_i \in \partial_s^A(V) : \mathrm{X_I} \subseteq \mathrm{U_i}, \mathrm{e_i} \notin \partial_\mathrm{s}^\mathrm{A}(\mathrm{X_I}), (\mathrm{X_O} \setminus \mathrm{X_I}) \cap \mathrm{U_i} = \emptyset\} && (\forall \mathrm{X} \in \mathcal{F}), \\
J_\mathrm{X} &:= \{e_i \in \partial_s^A(V) : \mathrm{X_I} \subseteq \mathrm{U_i}\} \setminus \mathrm{I_X} && (\forall \mathrm{X} \in \mathcal{F}), \\
b(\mathrm{X}) &:= \widetilde{m}(\mathrm{X_I}) - |\partial_\mathrm{s}^\mathrm{A}(\mathrm{X_I})| - \mathrm{r_1}(\mathrm{I_X} \cup \mathrm{J_X}) + \mathrm{r_1}(\mathrm{J_X}) && (\forall \mathrm{X} \in \mathcal{F}), \\
&:= +\infty && (\forall \mathrm{X} \notin \mathcal{F}).
\end{aligned}$$

Remark

$$I_\mathrm{X} \cup J_\mathrm{X} = \partial_s(P(\mathrm{X_I})).$$

$$b(X) := \widetilde{m}(X_I) - |\partial_s^A(X_I)| - (r_1(I_X \cup J_X) - r_1(J_X)) \qquad (\forall X \in \mathcal{F})$$

Lemma (implicitly in Bérczi, T. Király, Kobayashi)

Let $B \subseteq A$ for a given $D = (V + s, A)$. The following two conditions are equivalent:

$$|\partial_V^B(X)| \geq r_1(\partial_s^A(P_D(X))) - r_1(\partial_s^A(X)) \qquad (\forall X \subseteq V)$$

$$|\partial_V^B(X)| \geq r_1(I_X \cup J_X) - r_1(J_X) \qquad (\forall X \in \mathcal{F})$$

Theorem (K.)

In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M}_1)$ there exists an $\mathcal{M}_1$-reachability-based packing of s-arborescences iff

$$\varrho(X) = |\partial^A(X)| \geq r_1(\partial_s^A(P(X))) - r_1(\partial_s^A(X)) \ (\forall X \subseteq V).$$

$$b(X) := \widetilde{m}(X_I) - |\partial_s^A(X_I)| - (r_1(I_X \cup J_X) - r_1(J_X)) \qquad (\forall X \in \mathcal{F})$$

Lemma (implicitly in Bérczi, T. Király, Kobayashi)

Let $B \subseteq A$ for a given $D = (V + s, A)$. The following two conditions are equivalent:

$$|\partial_V^B(X)| \geq r_1(\partial_s^A(P_D(X))) - r_1(\partial_s^A(X)) \qquad (\forall X \subseteq V)$$

$$|\partial_V^B(X)| \geq r_1(I_X \cup J_X) - r_1(J_X) \qquad (\forall X \in \mathcal{F})$$

Theorem (K.)

In a matroid-rooted digraph $(D = (V + s, A), \mathcal{M}_1)$ there exists an $\mathcal{M}_1$-reachability-based packing of s-arborescences iff

$$\varrho(X) = |\partial^A(X)| \geq r_1(\partial_s^A(P(X))) - r_1(\partial_s^A(X)) \; (\forall X \subseteq V).$$

Lemma

b is an intersecting submodular bi-set function.

$$\mathcal{I}^* := \{B \subseteq A : i_B(X) \leq b(X)\ \forall X \in \mathcal{P}_2\}$$

forms the family of independent sets of a matroid $\mathcal{M}^*$ on A.

Theorem

Let $(D = (V + s, A), \mathcal{M}_1)$ be a matroid-rooted digraph with another matroid $\mathcal{M}_2 = \oplus_{v \in V} \mathcal{M}_v$ on A. Suppose that (A1), (A2) and (A3') are satisfied. Then $B \subseteq A$ is the arc set of an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences if and only if B is a common independent set of $\mathcal{M}_2$ and $\mathcal{M}^*$ of size $\widetilde{m}(V)$.

Lemma

b is an intersecting submodular bi-set function.

$$\mathcal{I}^* := \{B \subseteq A : i_B(\mathrm{X}) \le \mathrm{b}(\mathrm{X})\ \forall \mathrm{X} \in \mathcal{P}_2\}$$

forms the family of independent sets of a matroid $\mathcal{M}^*$ on A.

Theorem

Let $(D = (V + s, A), \mathcal{M}_1)$ be a matroid-rooted digraph with another matroid $\mathcal{M}_2 = \oplus_{v \in V} \mathcal{M}_v$ on A. Suppose that (A1), (A2) and (A3') are satisfied. Then $B \subseteq A$ is the arc set of an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences if and only if B is a common independent set of $\mathcal{M}_2$ and $\mathcal{M}^*$ of size $\widetilde{m}(V)$.

Theorem

Let $(D = (V + s, A), \mathcal{M}_1)$ be a matroid-rooted digraph with another matroid $\mathcal{M}_2 = \oplus_{v \in V} \mathcal{M}_v$ on A. There exists a polynomial algorithm to find an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences in D of minimum weight if D has at least one such packing.

Theorem

Let $(D = (V + s, A), \mathcal{M}_1)$ be a matroid-rooted digraph with another matroid $\mathcal{M}_2 = \oplus_{v \in V} \mathcal{M}_v$ on A. There exists an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences in D if and only if

$$r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial_s(P(X))) \text{ for all } X \subseteq V \text{ and } F \subseteq \partial_s(X).$$

Theorem

Let $(D = (V + s, A), \mathcal{M}_1)$ be a matroid-rooted digraph with another matroid $\mathcal{M}_2 = \oplus_{v \in V} \mathcal{M}_v$ on A. There exists a polynomial algorithm to find an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences in D of minimum weight if D has at least one such packing.

Theorem

Let $(D = (V + s, A), \mathcal{M}_1)$ be a matroid-rooted digraph with another matroid $\mathcal{M}_2 = \oplus_{v \in V} \mathcal{M}_v$ on A. There exists an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences in D if and only if

$$r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial_s(P(X))) \text{ for all } X \subseteq V \text{ and } F \subseteq \partial_s(X).$$

Theorem

Zoli still works on this topic.

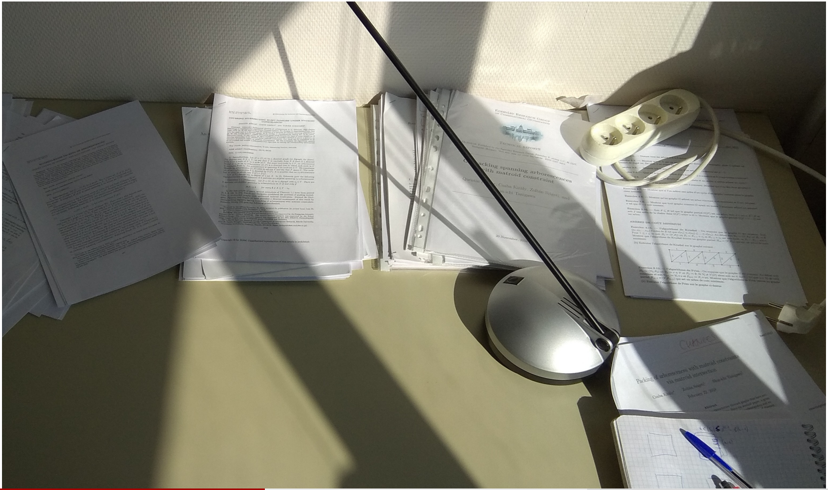
Proof.

□

Theorem

Zoli still works on this topic.

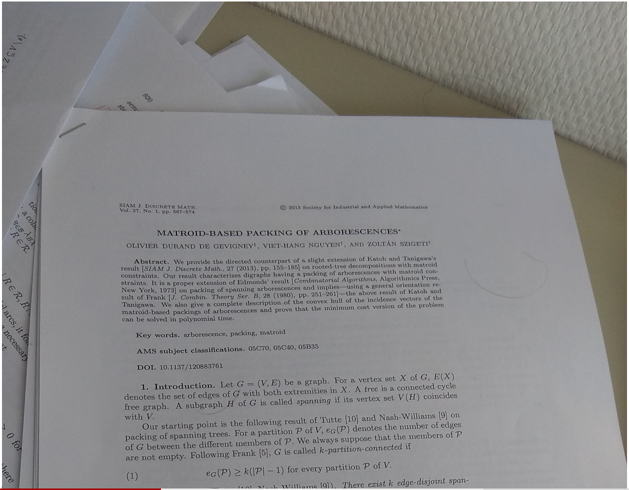
Proof.



Theorem

Zoli still works on this topic.

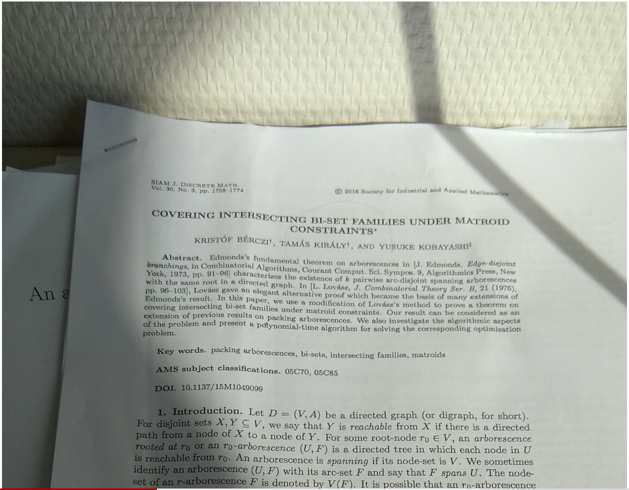
Proof.



Theorem

Zoli still works on this topic.

Proof.


SIAM J. DISCRETE MATH.
Vol. 30, No. 3, pp. 1758–1774

COVERING INTERSECTING BI-SET FAMILIES UNDER MATROID CONSTRAINTS*

KRISTÓF BÉRCZI†, TAMÁS KIRÁLY†, AND YUSUKE KOBAYASHI‡

Abstract. Edmonds's fundamental theorem on arborescences in [J. Edmonds, *Edge-disjoint branchings*, in Combinatorial Algorithms, Courant Comput. Sci. Sympos. 9, Algorithmics Press, New York, 1973, pp. 91–96] characterizes the existence of k pairwise arc-disjoint spanning arborescences with the same root in a directed graph. In [L. Lovász, *J. Combinatorial Theory Ser. B*, 21 (1976), pp. 96–103], Lovász gave an elegant alternative proof which became the basis of many extensions of Edmonds's result. In this paper, we use a modification of Lovász's method to prove a theorem on covering intersecting bi-set families under matroid constraints. Our result can be considered as an extension of previous results on packing arborescences. We also investigate the algorithmic aspects of the problem and present a polynomial-time algorithm for solving the corresponding optimization problem.

Key words. packing arborescences, bi-sets, intersecting families, matroids

AMS subject classifications. 05C70, 05C85

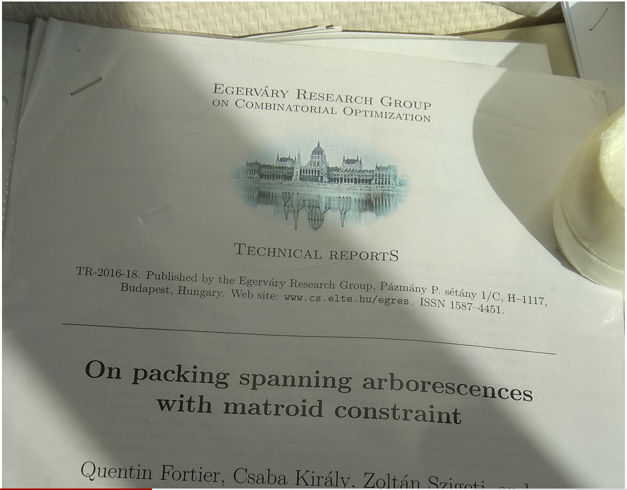
DOI. 10.1137/15M1049099

1. Introduction. Let $D = (V, A)$ be a directed graph (or digraph, for short). For disjoint sets $X, Y \subseteq V$, we say that Y is *reachable* from X if there is a directed path from a node of X to a node of Y. For some root-node $r_0 \in V$, an *arborescence rooted at* r_0 or an r_0*-arborescence* (U, F) is a directed tree in which each node in U is reachable from r_0. An arborescence is *spanning* if its node-set is V. We sometimes identify an arborescence (U, F) with its arc-set F and say that F *spans* U. The node-set of an r-arborescence F is denoted by $V(F)$. It is possible that an r_0-arborescence

Theorem

Zoli still works on this topic.

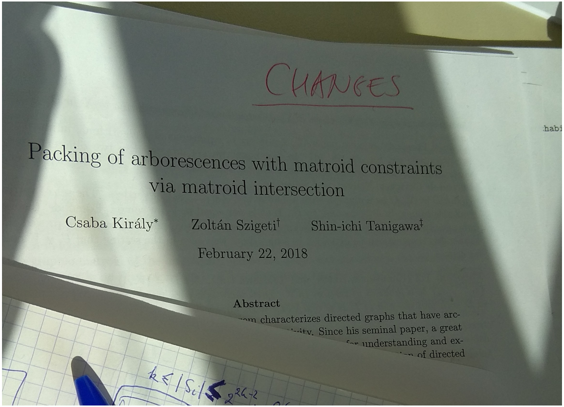
Proof.



Theorem

Zoli still works on this topic.

Proof.



□

Thank you for your attention!
Happy Birthday Zoli!