# A sequential game on matroids

Denis Cornaz

University Paris-Dauphine

# Plan

(Gourvès and Monnot 2013)

Given a matroid $M$ on $E$, let $A := \emptyset$ and $B := \emptyset$.

**While** $A \cup B \notin \mathcal{B}(M)$
Zoltan chooses $a \in E \setminus (A \cup B)$ so that $\{a\} \cup A \cup B \in \mathcal{I}(M)$
$A := A \cup \{a\}$
Then I choose $b \in E \setminus (A \cup B)$ so that $\{b\} \cup A \cup B \in \mathcal{I}(M)$
$B := B \cup \{b\}$.

Zoltan has an evaluation $w \in \mathbb{Z}^E$ and he wants to maximize $w(A)$ knowing

1. My evaluation, and
2. At each step, I choose the best possible element for me

(Gourvès and Monnot 2013)

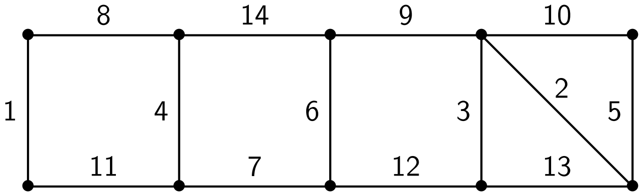
Given a matroid $M$ on $E$, let $A := \emptyset$ and $B := \emptyset$.

**While** $A \cup B \notin \mathcal{B}(M)$
Zoltan chooses $a \in E \setminus (A \cup B)$ so that $\{a\} \cup A \cup B \in \mathcal{I}(M)$
$A := A \cup \{a\}$
Then I choose $b \in E \setminus (A \cup B)$ so that $\{b\} \cup A \cup B \in \mathcal{I}(M)$
$B := B \cup \{b\}$.

Zoltan has an evaluation $w \in \mathbb{Z}^E$ and he wants to maximize $w(A)$ knowing

1. My evaluation, and
2. At each step, I choose the best possible element for me (unique)

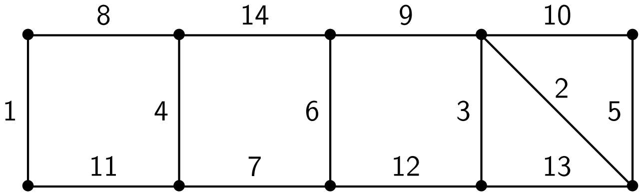
Graphic matroid on $E = \{1, \ldots, 14\}$




If I were alone I would take $B = \{1, \ldots, 9\}$, and Zoltan's evaluation is

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $e =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| $w_e =$ | 4 | 3 | 1 | 2 | 1 | 2 | 1 | 1 | 6 | 3 | 1 | 1 | 3 | 4 |

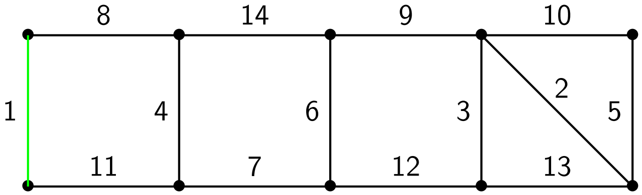
Graphic matroid on $E = \{1, \ldots, 14\}$




If I were alone I would take $B = \{1, \ldots, 9\}$, and Zoltan's evaluation is

| $e =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w_e =$ | 4 | 3 | 1 | 2 | 1 | 2 | 1 | 1 | 6 | 3 | 1 | 1 | 3 | 4 |

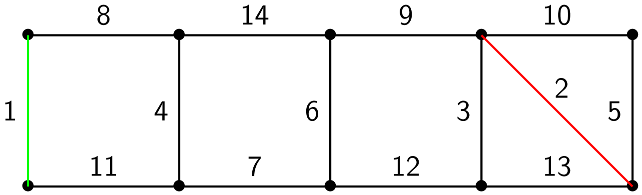
Graphic matroid on $E = \{1, \ldots, 14\}$




If I were alone I would take $B = \{1, \ldots, 9\}$, and Zoltan's evaluation is

| $e =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w_e =$ | 4 | 3 | 1 | 2 | 1 | 2 | 1 | 1 | 6 | 3 | 1 | 1 | 3 | 4 |

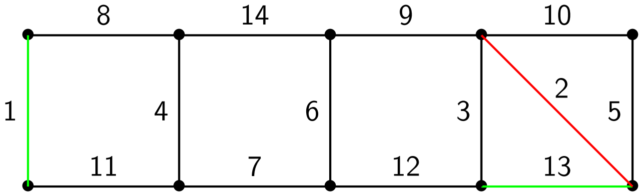
Graphic matroid on $E = \{1, \dots, 14\}$




If I were alone I would take $B = \{1, \dots, 9\}$, and Zoltan's evaluation is

| $e =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w_e =$ | 4 | 3 | 1 | 2 | 1 | 2 | 1 | 1 | 6 | 3 | 1 | 1 | 3 | 4 |

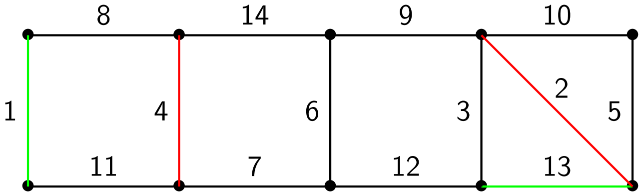
Graphic matroid on $E = \{1, \ldots, 14\}$




If I were alone I would take $B = \{1, \ldots, 9\}$, and Zoltan's evaluation is

| $e =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w_e =$ | 4 | 3 | 1 | 2 | 1 | 2 | 1 | 1 | 6 | 3 | 1 | 1 | 3 | 4 |

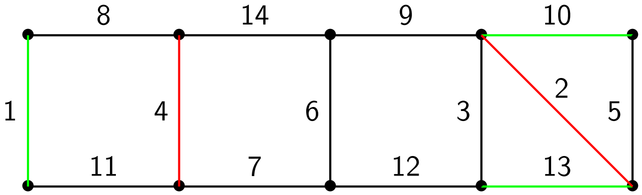
Graphic matroid on $E = \{1, \ldots, 14\}$




If I were alone I would take $B = \{1, \ldots, 9\}$, and Zoltan's evaluation is

| $e =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w_e =$ | 4 | 3 | 1 | 2 | 1 | 2 | 1 | 1 | 6 | 3 | 1 | 1 | 3 | 4 |

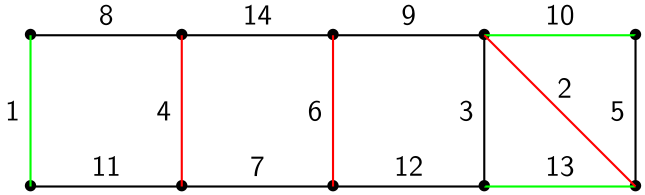
Graphic matroid on $E = \{1, \ldots, 14\}$




If I were alone I would take $B = \{1, \ldots, 9\}$, and Zoltan's evaluation is

| $e =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w_e =$ | 4 | 3 | 1 | 2 | 1 | 2 | 1 | 1 | 6 | 3 | 1 | 1 | 3 | 4 |

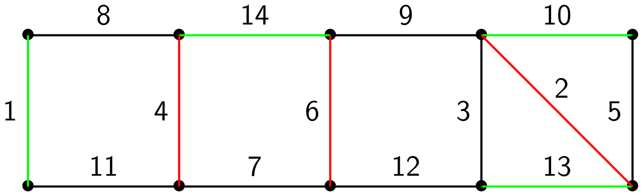
Graphic matroid on $E = \{1, \ldots, 14\}$




If I were alone I would take $B = \{1, \ldots, 9\}$, and Zoltan's evaluation is

| $e =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w_e =$ | 4 | 3 | 1 | 2 | 1 | 2 | 1 | 1 | 6 | 3 | 1 | 1 | 3 | 4 |

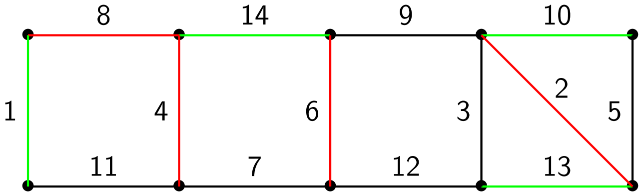
Graphic matroid on $E = \{1, \dots, 14\}$




If I were alone I would take $B = \{1, \dots, 9\}$, and Zoltan's evaluation is

| $e =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w_e =$ | 4 | 3 | 1 | 2 | 1 | 2 | 1 | 1 | 6 | 3 | 1 | 1 | 3 | 4 |

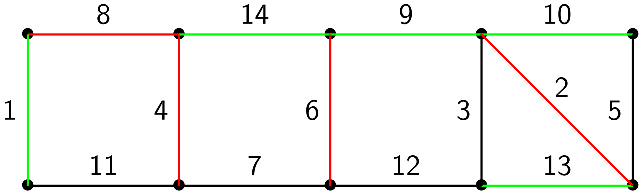
Graphic matroid on $E = \{1, \ldots, 14\}$




If I were alone I would take $B = \{1, \ldots, 9\}$, and Zoltan's evaluation is

| $e =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w_e =$ | 4 | 3 | 1 | 2 | 1 | 2 | 1 | 1 | 6 | 3 | 1 | 1 | 3 | 4 |

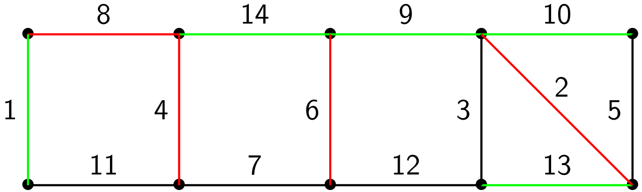
Graphic matroid on $E = \{1, \dots, 14\}$




If I were alone I would take $B = \{1, \dots, 9\}$, and Zoltan's evaluation is

| $e =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w_e =$ | 4 | 3 | 1 | 2 | 1 | 2 | 1 | 1 | 6 | 3 | 1 | 1 | 3 | 4 |

Graphic matroid on $E = \{1, \ldots, 14\}$ with base $B = \{1, \ldots, 9\}$



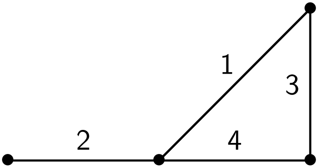

| $e =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w_e =$ | 4 | 3 | 1 | 2 | 1 | 2 | 1 | 1 | 6 | 3 | 1 | 1 | 3 | 4 |

Solution :

| Round | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Zoltan picks | 1 | 13 | 10 | 14 | 9 |
| I pick | 2 | 4 | 6 | 8 | . |

**Notation.** For any matroid $M$ with a base $B$, we denote by $\mathcal{A}(M, B)$ the set of Zoltan's possible subsets.

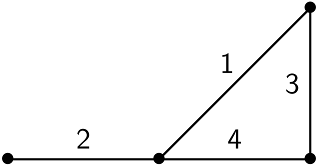
ex. $M$ graphic with $B = \{1, 2, 3\}$




$$\mathcal{A}(M, B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$$

**Notation.** For any matroid $M$ with a base $B$, we denote by $\mathcal{A}^{\uparrow}(M, B)$ the set of Zoltan's possible final sets.

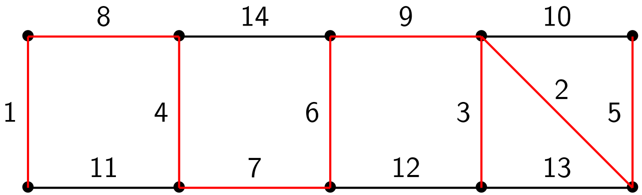
ex. $M$ graphic with $B = \{1, 2, 3\}$




$$\mathcal{A}^{\uparrow}(M, B) = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$$

**Lemma 1.** Given any matroid $M$ with a base $B = \{1, \dots, r(M)\}$, and $A \subseteq E$, testing if $A \in \mathcal{A}(M, B)$ is polynomial.

ex. $M$ graphic with $B = \{1, \dots, 9\}$

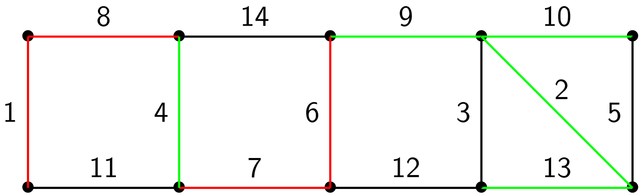



$\{1, 9, 10, 13, 14\}, \{2, 3, 5\} \in \mathcal{A}$

$\{1, 2\}, \{2, 5, 10\} \notin \mathcal{A}$

**Lemma 1.** Given any matroid $M$ with a base $B = \{1, \ldots, r(M)\}$, and $A \subseteq E$, testing if $A \in \mathcal{A}(M, B)$ is polynomial.

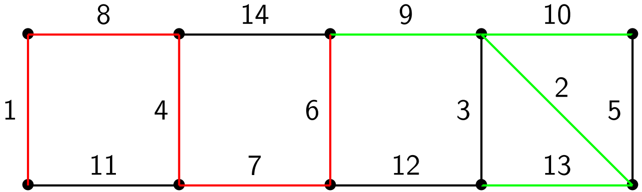
ex. $A = \{2, 4, 9, 10, 13\} \notin \mathcal{A}$




| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $B \ni$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $B' \ni$ | 1 | | | | | 6 | 7 | 8 | |
| $B \setminus B' \ni$ | | $e_1$ | $e_2$ | $e_3$ | $e_4$ | | | | $e_5$ |
| $e_j - j + 1 =$ | | 2 | 2 | 2 | 2 | | | | 5 |

**Lemma 1.** Given any matroid $M$ with a base $B = \{1, \dots, r(M)\}$, and $A \subseteq E$, testing if $A \in \mathcal{A}(M, B)$ is polynomial.

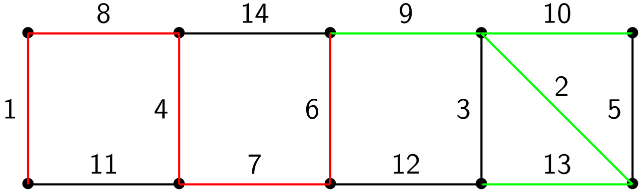
ex. $A = \{2, 9, 10, 13\} \in \mathcal{A}$




| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $B \ni$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $B' \ni$ | 1 | | | 4 | | 6 | 7 | 8 | |
| $B \setminus B' \ni$ | | $e_1$ | $e_2$ | | $e_3$ | | | | $e_4$ |
| $e_j - j + 1 =$ | | 2 | 2 | | 3 | | | | 6 |

**Lemma 1.** Given any matroid $M$ with a base $B = \{1, \dots, r(M)\}$, and $A \subseteq E$, testing if $A \in \mathcal{A}(M, B)$ is polynomial.

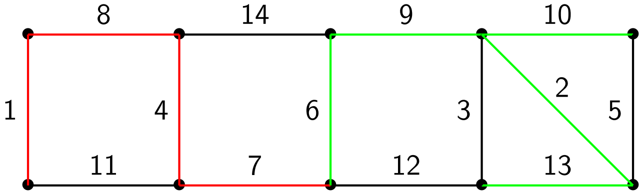
ex. $A = \{2, 9, 10, 13\} \in \mathcal{A}$




| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $B \ni$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $B' \ni$ | 1 | | | 4 | | 6 | 7 | 8 | |
| $B \setminus B' \ni$ | | $e_1$ | $e_2$ | | $e_3$ | | | | $e_4$ |
| $e_j - j + 1 =$ | | 2 | 2 | | 3 | | | | 6 |
| $F_j =$ | | $\{2\}$ | $\{2, 13\}$ | | $\{2, 10\}$ | | | | |

**Lemma 1.** Given any matroid $M$ with a base $B = \{1, \dots, r(M)\}$, and $A \subseteq E$, testing if $A \in \mathcal{A}(M, B)$ is polynomial.

ex. $A = \{2, 6, 9, 10, 13\} \notin \mathcal{A}$




| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $B \ni$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $B' \ni$ | 1 | | | 4 | | | 7 | 8 | |
| $B \setminus B' \ni$ | | $e_1$ | $e_2$ | | $e_3$ | $e_4$ | | | $e_5$ |
| $e_j - j + 1 =$ | | 2 | 2 | | 3 | 3 | | | 5 |
| $F_j =$ | | $\{2\}$ | $\{2, 13\}$ | | $\{2, 10\}$ | $\{6\}$ | | | |

**Lemma 1.** Given any matroid $M$ with a base $B = \{1, \dots, r(M)\}$, and $A \subseteq E$, testing if $A \in \mathcal{A}(M, B)$ is polynomial.

**Pf.** Let $B'$ be my prefered base in $M/A$, let $B \setminus B' = \{e_1, \dots, e_k\}$ (sorted), and let

$$F_j := \begin{cases} \{e_j\} & \text{if } e_j \in A \\ C_j \cap A & \text{otherwise (circuit } C_j \subseteq \{e_j\} \cup A \cup B) \end{cases}$$

Then

$$(a_1, b_1), \dots, (a_k, b_k) \text{ is feasible} \iff F_j \subseteq \{a_1, \dots, a_{e_j - j + 1}\} \quad (\forall j)$$

$$A \in \mathcal{A} \iff \left| \bigcup_{1 \le i \le j} F_i \right| \le e_j - j + 1 \quad (\forall j)$$

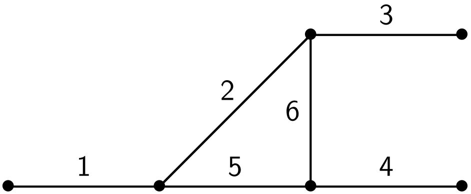
**Lemma 2.** Given any matroid $M$ with a base $B = \{1, \dots, r(M)\}$, and $A \in \mathcal{A}^{\uparrow}(M, B)$ then

$$\forall x \in E \setminus A, \exists y \in A : \; A \cup \{x\} \setminus \{y\} \in \mathcal{A}^{\uparrow}(M, B)$$

**Lemma 2.** Given any matroid $M$ with a base $B = \{1, \ldots, r(M)\}$, and $A \in \mathcal{A}^{\uparrow}(M, B)$ then

$$\forall x \in E \setminus A, \exists y \in A : \ A \cup \{x\} \setminus \{y\} \in \mathcal{A}^{\uparrow}(M, B)$$

ex. $M$ graphic with $B = \{1, 2, 3, 4, 5\}$




| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{A}^{\uparrow} \not\ni$ | 123 | 124 | 125 | 126 | 134 | | | | | 156 |
| $\mathcal{A}^{\uparrow} \ni$ | | | | | | 135 | 136 | 145 | 146 | |
| $\mathcal{A}^{\uparrow} \not\ni$ | 234 | | | | | 256 | | | 356 | |
| $\mathcal{A}^{\uparrow} \ni$ | | 235 | 236 | 245 | 246 | | 345 | 346 | | 456 |

**Lemma 2.** Given any matroid $M$ with a base $B = \{1, \ldots, r(M)\}$, and $A \in \mathcal{A}^{\uparrow}(M, B)$ then

$$\forall x \in E \setminus A, \exists y \in A : \; A \cup \{x\} \setminus \{y\} \in \mathcal{A}^{\uparrow}(M, B)$$

**Pf.** Zoltan and I played $(a_1, b_1), \ldots, (a_m, b_m)$. Let $x \in E \setminus A$ and let $i$ minimum so that Zoltan cannot take $x$ if he plays

$$(a_1, b_1), \ldots, (a_i, \ldots$$

Apply circuits axiom to show that, if Zoltan plays

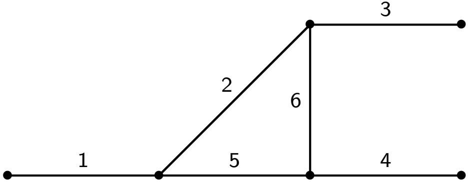
$$(a_1, b_1), \ldots, (x, \ldots$$

and then always choosing the possible $a_j$ with $j$ minimum, then he will have $A \cup \{x\} \setminus \{a_k\}$ (for some $k \geq i$).

**Remark 1.** In general, even if $M$ is graphic, $\mathcal{A}^{\uparrow}(M, B)$ is not the set of basis of a matroid.

**Remark 1.** In general, even if $M$ is graphic, $\mathcal{A}^{\uparrow}(M, B)$ is not the set of basis of a matroid.

ex. $M$ graphic with $B = \{1, 2, 3, 4, 5\}$

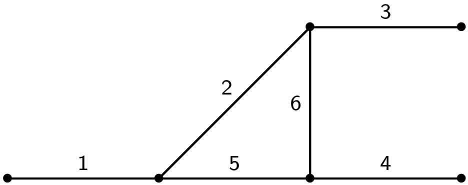



| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{A}^{\uparrow} \not\ni$ | 123 | 124 | 125 | 126 | 134 | | | | | 156 |
| $\mathcal{A}^{\uparrow} \ni$ | | | | | | 135 | 136 | 145 | 146 | |
| $\mathcal{A}^{\uparrow} \not\ni$ | 234 | | | | | 256 | | | 356 | |
| $\mathcal{A}^{\uparrow} \ni$ | | 235 | 236 | 245 | 246 | | 345 | 346 | | 456 |

$$A_1 = \{2, 3, 5\} \text{ and } A_2 = \{4, 5, 6\}$$

**Remark 1.** In general, even if $M$ is graphic, $\mathcal{A}^{\uparrow}(M, B)$ is not the set of basis of a matroid.

ex. $M$ graphic with $B = \{1, 2, 3, 4, 5\}$

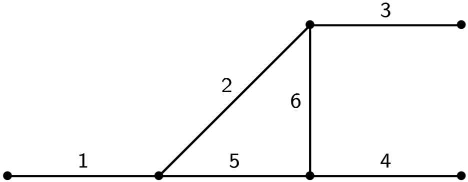



| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{A}^{\uparrow} \not\ni$ | 123 | 124 | 125 | 126 | 134 | | | | | 156 |
| $\mathcal{A}^{\uparrow} \ni$ | | | | | | 135 | 136 | 145 | 146 | |
| $\mathcal{A}^{\uparrow} \not\ni$ | 234 | | | | | 256 | | | 356 | |
| $\mathcal{A}^{\uparrow} \ni$ | | 235 | 236 | 245 | 246 | | 345 | 346 | | 456 |

$$A_1 = \{2, 3, 5\} \text{ and } A_2' = \{5, 6\}$$

**Remark 1.** In general, even if $M$ is graphic, $\mathcal{A}^{\uparrow}(M, B)$ is not the set of basis of a matroid.

ex. $M$ graphic with $B = \{1, 2, 3, 4, 5\}$

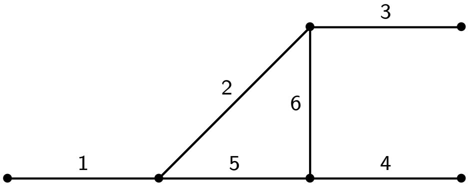



| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{A}^{\uparrow} \not\ni$ | 123 | 124 | 125 | 126 | 134 | | | | | 156 |
| $\mathcal{A}^{\uparrow} \ni$ | | | | | | 135 | 136 | 145 | 146 | |
| $\mathcal{A}^{\uparrow} \not\ni$ | 234 | | | | | 256 | | | 356 | |
| $\mathcal{A}^{\uparrow} \ni$ | | 235 | 236 | 245 | 246 | | 345 | 346 | | 456 |

$$C_1 = \{2, 5, 6\} \text{ and } C_2 = \{3, 5, 6\}$$

**Remark 1.** In general, even if $M$ is graphic, $\mathcal{A}^{\uparrow}(M, B)$ is not the set of basis of a matroid.

ex. $M$ graphic with $B = \{1, 2, 3, 4, 5\}$




| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{A}^{\uparrow} \not\ni$ | 123 | 124 | 125 | 126 | 134 | | | | | 156 |
| $\mathcal{A}^{\uparrow} \ni$ | | | | | | 135 | 136 | 145 | 146 | |
| $\mathcal{A}^{\uparrow} \not\ni$ | 234 | | | | | 256 | | | 356 | |
| $\mathcal{A}^{\uparrow} \ni$ | | 235 | 236 | 245 | 246 | | 345 | 346 | | 456 |

$$f(2356) + f(56) > f(256) + f(356)$$

**Remark 2.** If $M$ is free and $B = \{1, \ldots, r\}$, then $\mathcal{A}(M, B)$ is the set of bases of a laminar matroid with rank function satisfying

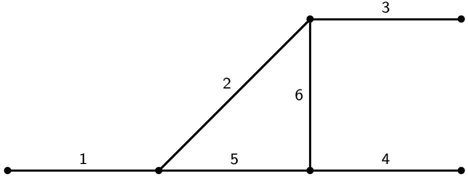
$$r(\{1, \ldots, 2i\}) \leq i$$

**Remark 2.** If $M$ is free and $B = \{1, \ldots, r\}$, then $\mathcal{A}(M, B)$ is the set of bases of a laminar matroid with rank function satisfying

$$r(\{1, \ldots, 2i\}) \leq i$$

However, it is not, in general, the intersection of $M$ and this laminar matroid.

ex. 356 in

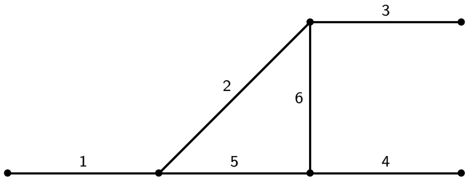


**Remark 2.** If $M$ is free and $B = \{1, \ldots, r\}$, then $\mathcal{A}(M, B)$ is the set of bases of a laminar matroid with rank function satisfying

$$r(\{1, \ldots, 2i\}) \leq i$$

In fact, it is not, in general, the intersection of two matroids.

ex. $\mathcal{C} = \{12, 156, 256\} \cup \{12, 134, 234\} \cup \{356\}$



# Open questions

1. Can we find Zoltan's best strategy in polynomial time ?
2. What kind of things are Zoltan's sets $\mathcal{A}(M, B)$ ?