Connected Orientations

Zoltán Király

Department of Computer Science, Eötvös Loránd University, Budapest

Zoli Szigeti is 50!

Grenoble, 20 March 2018

Happy Birthday, Zoli!

KöMaL, 1983.

Tokyo, Zojoji Temple, 2011

After the first NII Shonan Meeting, 3 weeks before the earthquake

Preliminary

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- Now I will speak about only some joint results regarding orientations.

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Theorem ("Weak" Nash-Williams)

An undirected graph G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.

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- If G is Eulerian, then every Eulerian orientation is well-balanced.

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Claim (Nash-Williams)

For an arbitrary subgraph H of an undirected graph G, there exists a best-balanced orientation of H that can be extended to a best-balanced orientation of G.

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Theorem ("Strong pairing" theorem, Nash-Williams)

For every graph there is a pairing M whose every orientation is good.

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- Using his guideline, I shortly presented a proof for the missing part.

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Theorem (Zoli+Zoli)

Let $G_1, G_2, \ldots, G_k$ be edge-disjoint subgraphs of G. Then G has a best-balanced orientation $\vec{G}$, such that the inherited orientation of each G_i is also best-balanced.

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- An Eulerian graph G has an Eulerian orientation $\vec{G}$ such that $\vec{G} - v$ is k-arc-connected for all $v \in V$ if and only if $G - v$ is $2k$-edge-connected for all $v \in V$.

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- Let v be a vertex of graph G. G has a k-arc-connected orientation $\vec{G}$ such that $\vec{G} - v$ is $(k-1)$-arc-connected if and only if G is $2k$-edge-connected and $G - v$ is $(2k-2)$-edge-connected.

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- An undirected graph G on more than k vertices has a k-vertex-connected orientation if and only if for every vertex set $X \subseteq V$ with $|X| \leq k,\ G - X$ is $(2k - 2|X|)$-edge-connected.

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- Counterexample was given in 2012 by Olivier Durand de Gevigney.

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- I give now only some examples.

NP-completeness

- Problems MinCostWellBalanced and MinCostBestBalanced: Given a graph G, non-negative integer costs for the two possible orientations of each edge, and an integer bound K. Is there a well-balanced (best-balanced) orientation of G with total cost at most K?

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- Problems BoundedWellBalanced and BoundedBestBalanced: Given a graph G, $\ell, u : V \to \mathbb{Z}_+$ bounds with $\ell \leq u$. Is there a well-balanced (best-balanced) orientation $\vec{G}$ of G with $\ell(v) \leq \delta_{\vec{G}}(v) \leq u(v)$ for every $v \in V$?

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- **These problems are also NP-complete.**

NP-completeness (contd)

- Deciding whether two Eulerian graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ have Eulerian orientations that agree on the common edges $E_1 \cap E_2$, is *NP*-complete.

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- Let $G_1 \subseteq G_2 \subseteq G_3$. Is there an orientation $\vec{G}_3$ such that the inherited orientation of $\vec{G}_i$ is i-arc-connected for all i?

Happy Birthday, Zoli!

Tokyo, 2011

Budapest, 2009, “András Frank is 60”

Happy Birthday, Zoli!

Exactly 50 candles, try to count them!